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SOME CONTRIBUTIONS TO NONLINEAR SYSTEMS OF FUNCTIONAL INTEGRAL EQUATIONS

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Nonlinear evolution equations, i.e., partial differential equations with time t as one of the independent variables, arise not only from many fields of mathematics, but also from other branches of science such as physics, mechanics and material science [40]. For example, Navier-Stokes and Euler equations from fluid mechanics, nonlinear reaction-diffusion equations from heat transfers and biological sciences, nonlinear Klein-Gorden equations and nonlinear Schrodinger equations from quantum mechanics and Cahn-Hilliard equations from material science, to name just a few, are special examples of nonlinear evolution equations. Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences. Many such problems in the fields of ordinary and partial differential equations can be recasted as integral and integrodifferential equations. Several existence and uniqueness results can be derived from the corresponding results of integral equations. Such results can be obtained by applying a variety of fixed point theorems as illustrated in [18, 20–23, 30, 31]. For example, Schauder's fixed point theorem is helpful in asserting the existence of solutions of integrodifferential equations whereas The Banach fixed point theorem is an important source of existence and uniqueness theorem in different branches of analysis. Recently, Schaefer fixed point theorem is also used to prove the existence of solutions of various types of functional differential equations.

The evolution of a physical system in time is described by an initial value problem of the form

$$\frac{du}{dt} = Au(t), \quad t \geq 0 \quad \text{and} \quad u(0) = u_0$$

where $A : D(A) \longrightarrow X$ is a linear operator with domain $D(A) \subset X$, X being a Banach space, $u : [0, \infty) \longrightarrow X$ and $u_0 \in D(A)$. Here A does not depend on time t. This means that the underlying mechanism does not depend on time. The above initial value problem leads to the concept of one parameter semigroup $\{T(t)\}, t \geq 0$, of bounded linear operators on a Banach space X.

The theory of semigroups of bounded linear operators is closely related to the solutions of differential and integrodifferential equations in Banach spaces [32]. This theory developed from the end of 1940s to the 1960s, and it is still very useful nowadays. It developed quite rapidly since the discovery of the generation theorem by Hille and Yosida in 1948. In recent years, this theory has been applied to a large class of nonlinear differential equations in Banach spaces. Using the method of semigroups, existence and uniqueness of mild, strong and classical solutions of evolution equations have been discussed in Pazy [36]. Usually, each "well-posed" linear autonomous initial value problem give rise to a semigroup of bounded linear operators. By now, it is an extensive mathematical subject with substantial applications to many fields of analysis.

The work on nonlocal initial value problem was initiated by Byszewski. Motivated by physical applications, Byszewski, Acka et al studied a variety of nonlocal Cauchy problems, see [6, 11, 12, 25, 35, 37, 43, 44]. The importance of nonlocal conditions in different fields has been discussed in [10], [14] and the references therein. For example, in [14] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$g(x) = \sum_{i=0}^{p} C_i x(t_i),$$

where $C_i, i = 0, 1, ..., p$ are given constants and $0 < t_1 < t_2 < \cdots < t_p < b$. In this case the above equation allows the additional measurement at $t_i, i = 0, 1, ..., p$. The notion of 'nonlocal condition' was introduced to extend the study of the classical initial value problems.

The study of Cauchy problems with nonlocal conditions is of great significance. Many authors have investigated the problem of nonlocal initial conditions for different classes of abstract differential equations in Banach spaces, see ([2],[10],[33],[41]). The nonlocal Cauchy problem for functional differential equations with delay was also studied by Byszewski [11],[12]. In particular the theory of functional differential equations with nonlocal conditions has been the

area of interest of many authors, see ([6],[11],[12]). This type of nonlocal problems have been studied by employing various fixed point theorems. In [6] and [11], authors have used the Banach fixed point principle whereas in [12], Schauder's fixed point theorem is employed to derive the existence results.

Neutral differential equations arise in many areas of applied mathematics and for this reason, this type of equation have received much attention in recent years see [1, 4, 5, 8, 9, 13, 38, 39]. Good guide to the literature for neutral functional differential equations are, the related chapter in the book of Hale [26], the Hale and Lunel book [27] and the references cited therein. Hernandez and Henriquez in [28],[29] established the existence results for different types of partial neutral functional differential equations with nonlocal conditions. Tremendous work has also been done on various types of abstract nonlinear functional integral and integrodifferential equations[15–19]. Several papers have appeared for the existence and controllability of solutions of the nonlinear second-order neutral functional differential equations in Banach spaces [3, 7]. In many cases it is advantageous to treat second order abstract differential equations directly rather than to convert them to first order systems. A useful machinery for the study of second order equations is the theory of strongly continuous cosine family [42].

After studying the available literature, the author strongly feels that the field of abstract functional integrodifferential equations with nonlocal condition of more general type, yet awaits for its further development. This motivates the author to study nonlocal functional integrodifferential equations with nonlocal condition of more general type.
Our analysis in the present thesis entitled **"Some contributions to nonlinear systems of functional integral equations"** is based on Leray-Schauder Alternative, semigroup theory, Hausdorff's measure of noncompactness, Darbo-Sadovskii's and Sadovskii's fixed point theorem, fractional power of operators and the theory of cosine and sine family. The thesis is conveniently divided into five chapters. Chapter 1 is an introductory chapter. In the next four chapters

we introduce some important methods for the investigation of results pertaining to existence, uniqueness, continuous dependence and controllability of solutions of various first and second order nonlinear Volterra-Fredholm functional integrodifferential equations with nonlocal conditions in Banach spaces.

Chapter 2 discusses the existences of mild solutions of the following nonlinear Volterra-Fredholm functional integrodifferential equations with nonlocal conditions in a general Banach space :

$$x^{'}(t) = f\bigg(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^T b(t,s)k(s,x_s)ds\bigg), \quad t \in [0,T], \tag{1}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t), \quad t \in [-r, 0], \tag{2}$$

$$\big(r(t)x^{'}(t)\big)^{'} = f\bigg(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^T b(t,s)k(s,x_s)ds\bigg), \quad t \in [0,T], \tag{3}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t), \quad t \in [-r, 0], \quad x^{'}(0) = 0 \tag{4}$$

$$x^{'}(t) + Ax(t) = f\bigg(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^T b(t,s)k(s,x_s)ds\bigg), \quad t \in [0,T], \tag{5}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t), \quad t \in [-r, 0], \tag{6}$$

$$\frac{d}{dt}[x(t) - w(t, x_t)] + Ax(t) = f\bigg(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^T b(t,s)k(s,x_s)ds\bigg), \quad t \in [0,T], \tag{7}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t), \quad t \in [-r, 0], \tag{8}$$

where $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X, $0 < t_1 < ... <$

$t_p \leq T$, $p \in N$, $f : [0,T] \times C \times X \times X \to X$, $a, b : [0,T] \times [0,T] \to \mathbb{R}$, $w, h, k : [0,T] \times C \to X$ are continuous functions, $g : C^p \to C$ is given, ϕ is a given element of C and $r(t)$ is a real valued, positive and sufficiently smooth function defined on [0,T]. In this chapter our analysis is based on the Leray-Schauder Alternative given in [24] which rely on the priori bounds of solution and the semigroup theory. The advantage of using this fixed point theorem lies in the fact that we do not claim conditions which imply $FU \subset U$, where U is a set and F is an operator [34]. We also present examples to illustrate the application of our results.

In Chapter 3 we study the existences, uniqueness and continuous dependence of solutions of the nonlocal problem (5)-(6) and the neutral functional nonlocal problem (7)-(8), in a Banach space, where $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)$ in X. The method of Hausdorff's measure of noncompactness and Darbo-Sadovskii fixed point theorem are used to establish our results. An application related to controllabilty is provided to illustrate the theory.

Chapter 4 is devoted to the existence of mild and strong solutions of the abstract nonlinear mixed functional integrodifferential equation (7) with nonlocal conditions (8), where $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)$ in X. Our approach is based on Sadovskii's fixed point theorem and the theory of fractional power of operators.

Chapter 5 is concerned with the existence of mild solution of the following nonlinear second order mixed functional integrodifferential equation with nonlocal condition in Banach space

$$\frac{d}{dt}[x'(t) - w(t, x_t)] + Ax(t) = f\Bigg(t, x_t, \int_0^t a(t,s)h(s,x_s)ds, \int_0^T b(t,s)k(s,x_s)ds\Bigg), \quad t \in [0,T], \tag{9}$$

$$x(t) + \big(g(x_{t_1}, ..., x_{t_p})\big)(t) = \phi(t), \quad t \in [-r, 0], \qquad x'(0) = \xi \in X, \tag{10}$$

where $0 < t_1 < ... < t_p \leq T$, $p \in N$, $f : [0,T] \times C \times X \times X \to X$, $a, b : [0,T] \times [0,T] \to \mathbb{R}$, $w, h, k : [0,T] \times C \to X$ are continuous functions, $g : C^p \to C$ is given, ϕ is a given element of C. $-A$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$ of bounded linear operators in X. The results are derived by employing two different techniques namely the Darbo-Sadovskii's fixed point theorem with Hausdorff's measure of noncompactness and the Leray Schauder Alternative. We also make a comparison between both the techniques and highlight the advantages of one method over the other.

We note that, in the last few years the scientific and engineering problems involving fractional calculus are studied in very large number and are still growing, and perhaps fractional calculus will be the calculus of the twenty-first century. In the last two decades, lots of literature in the form of research papers and books has been published showing applications of this useful branch of mathematics. It has been found that various interdisciplinary applications can be elegantly and more accurately modeled with the help of fractional derivatives. So there is a wide scope for applying various fixed-point techniques to solve fractional differential and integral equations and this will be an active research undertaking in future.

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